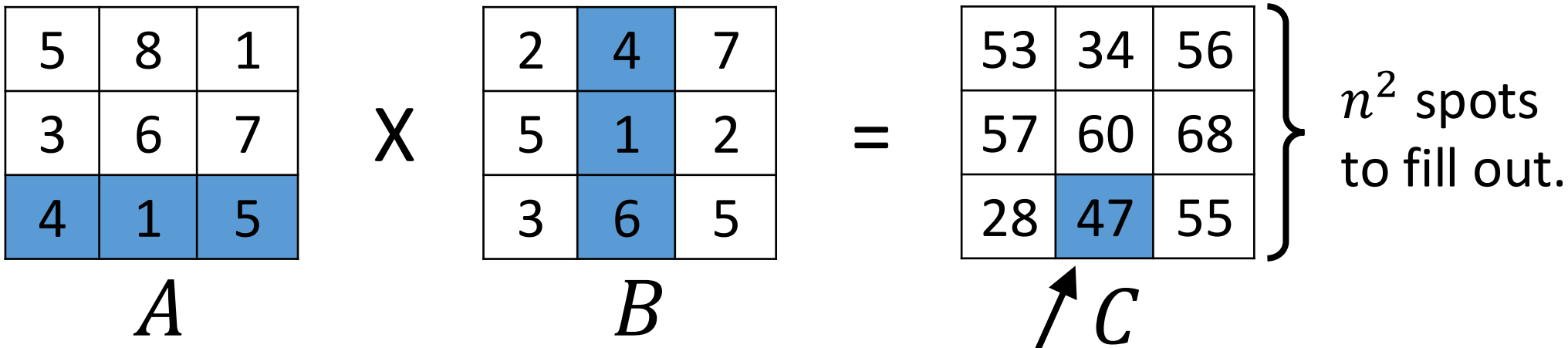
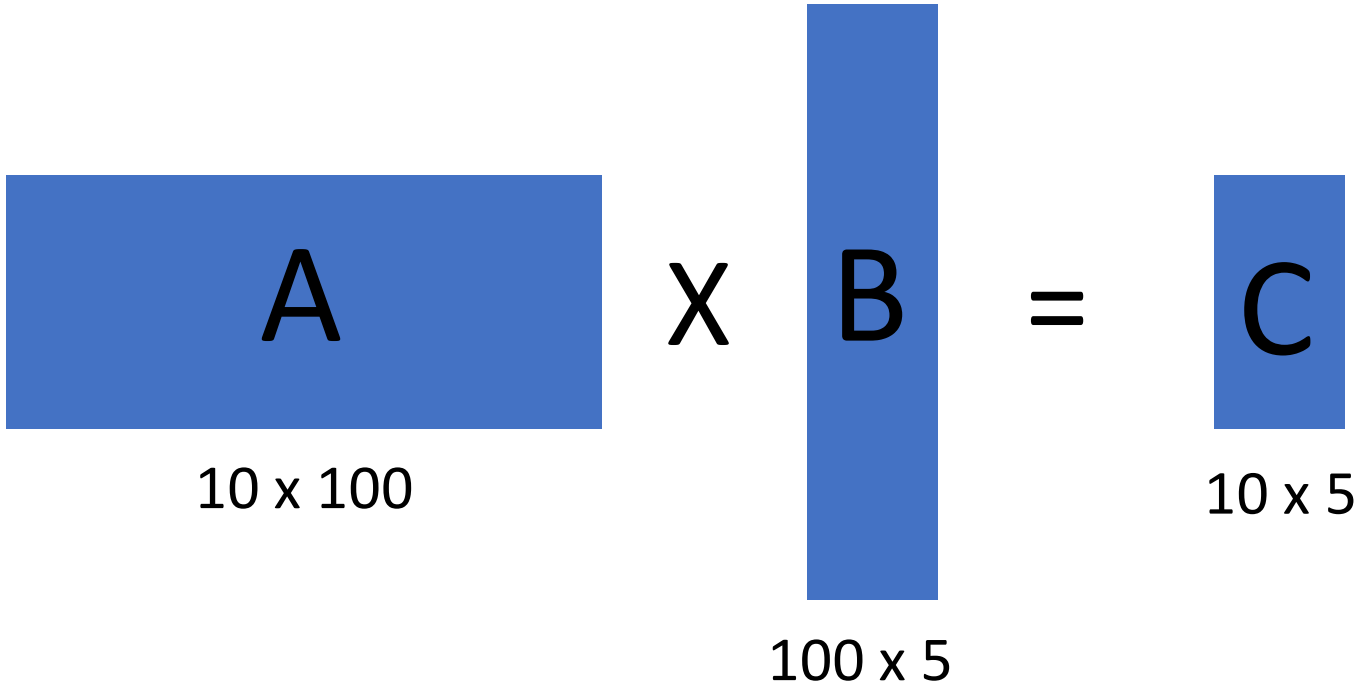
Dynamic Programming
CSCI 432

Square Matrix Multiplication

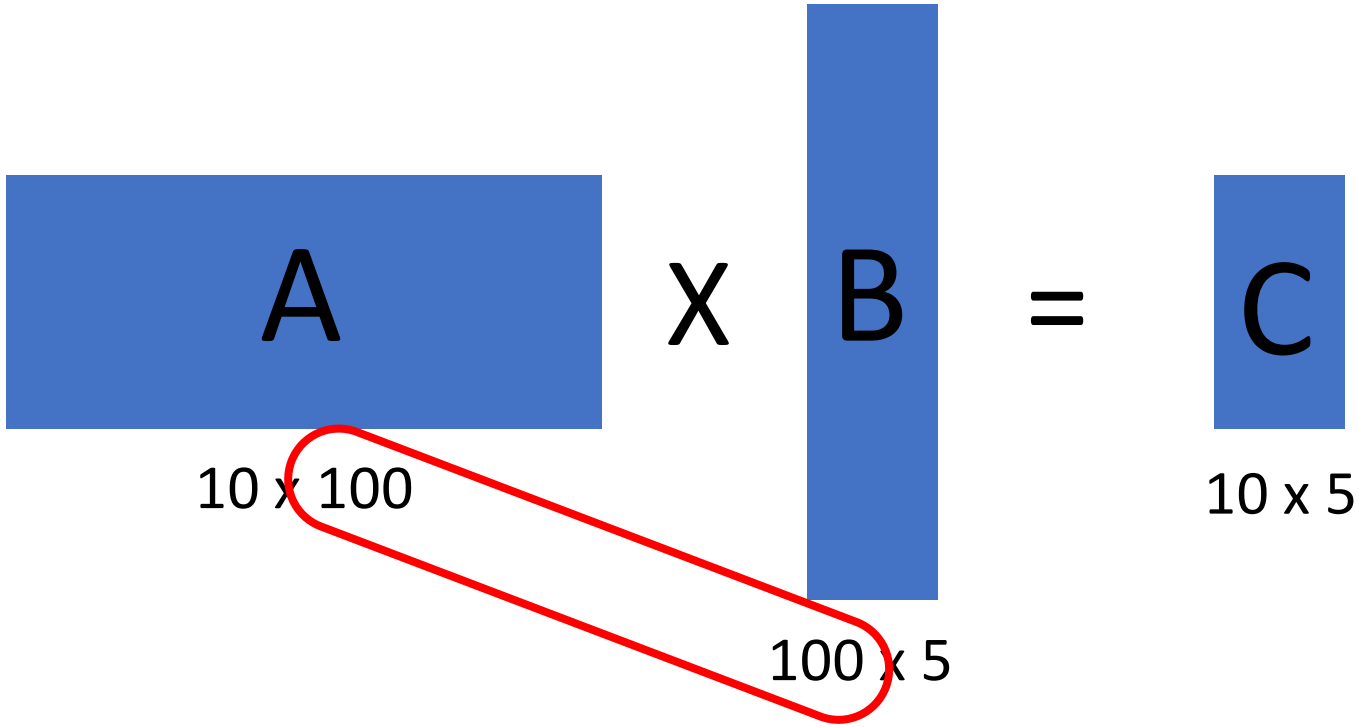


Thus, $O(n^3)$ total running time.

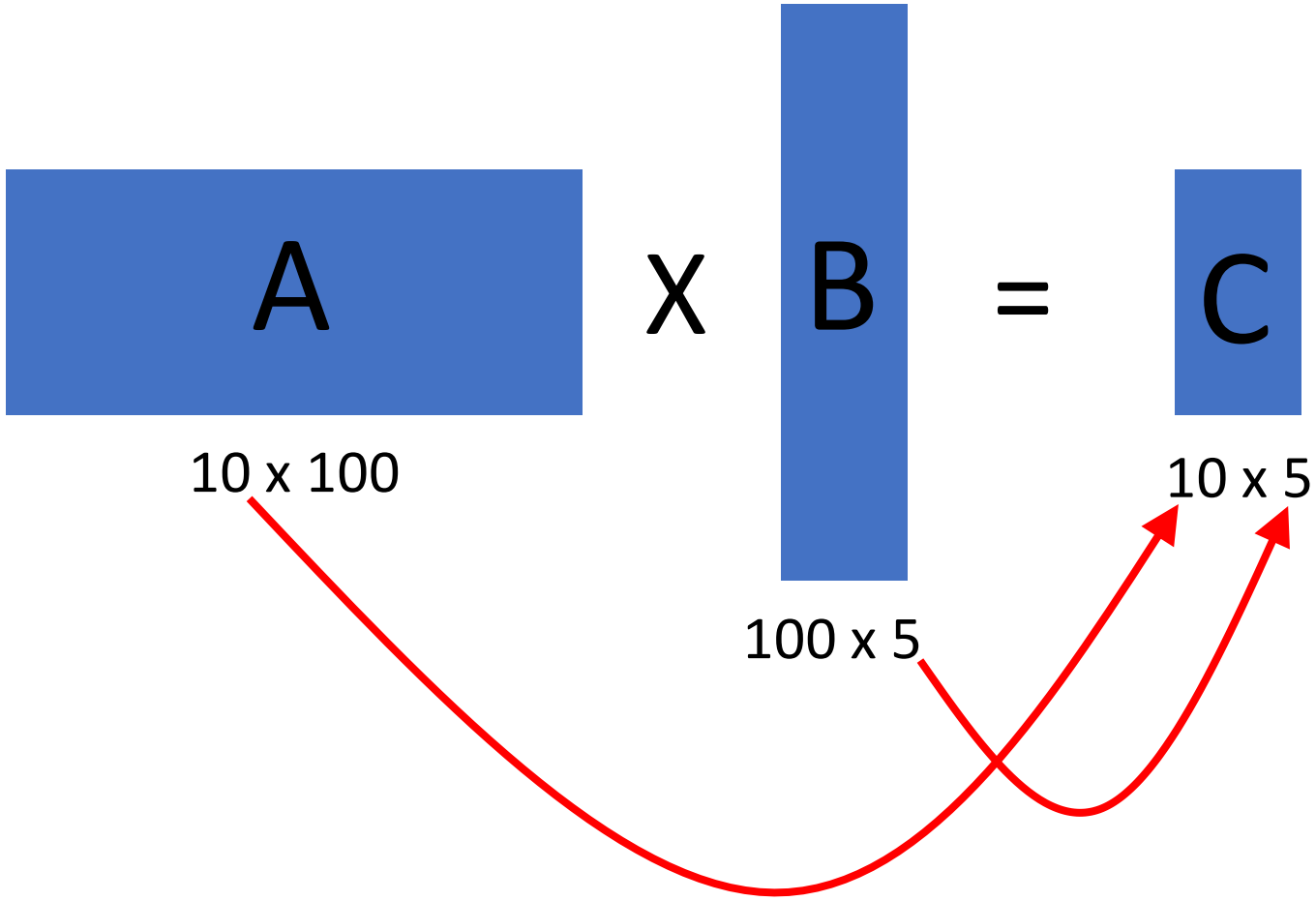
Non-Square Matrix Multiplication Review



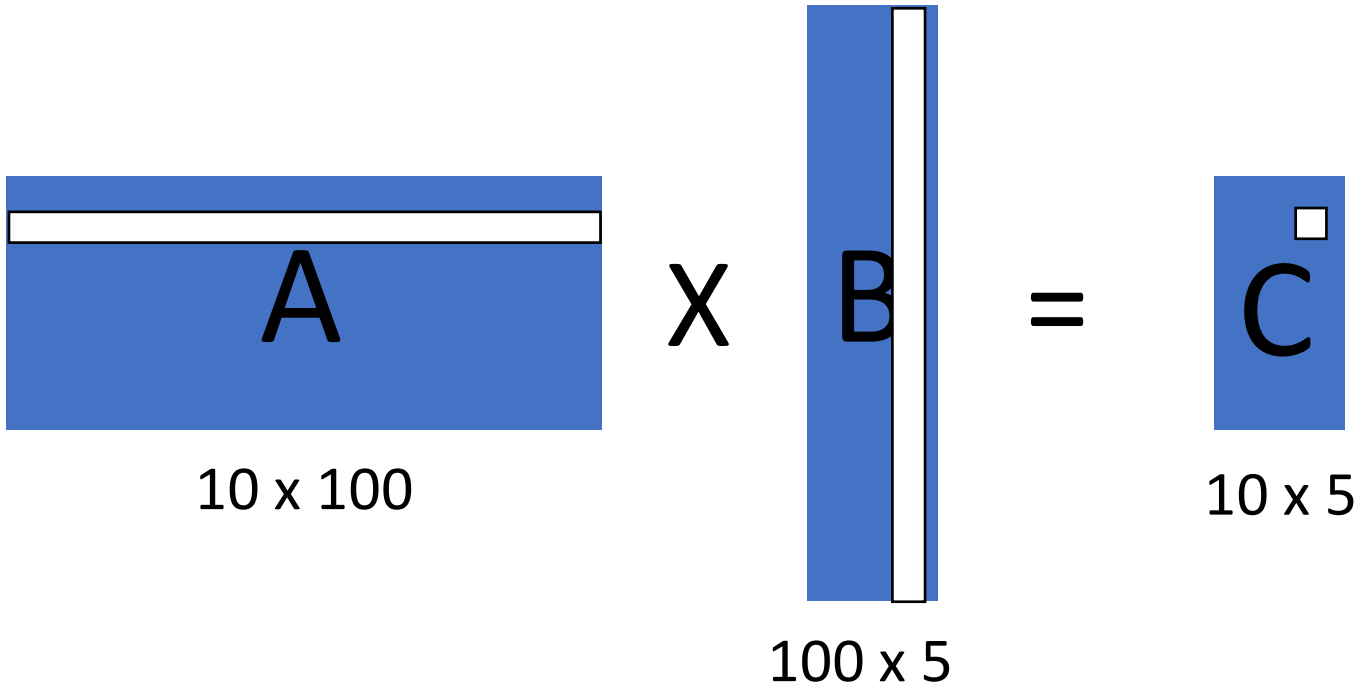
Non-Square Matrix Multiplication Review



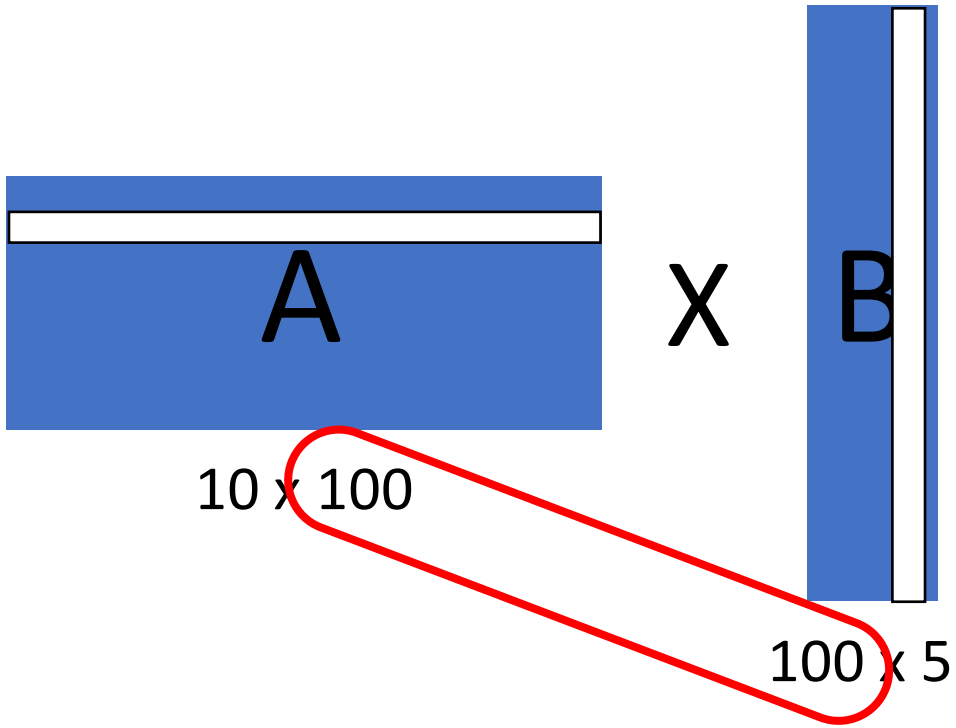
Non-Square Matrix Multiplication Review



Non-Square Matrix Multiplication Review



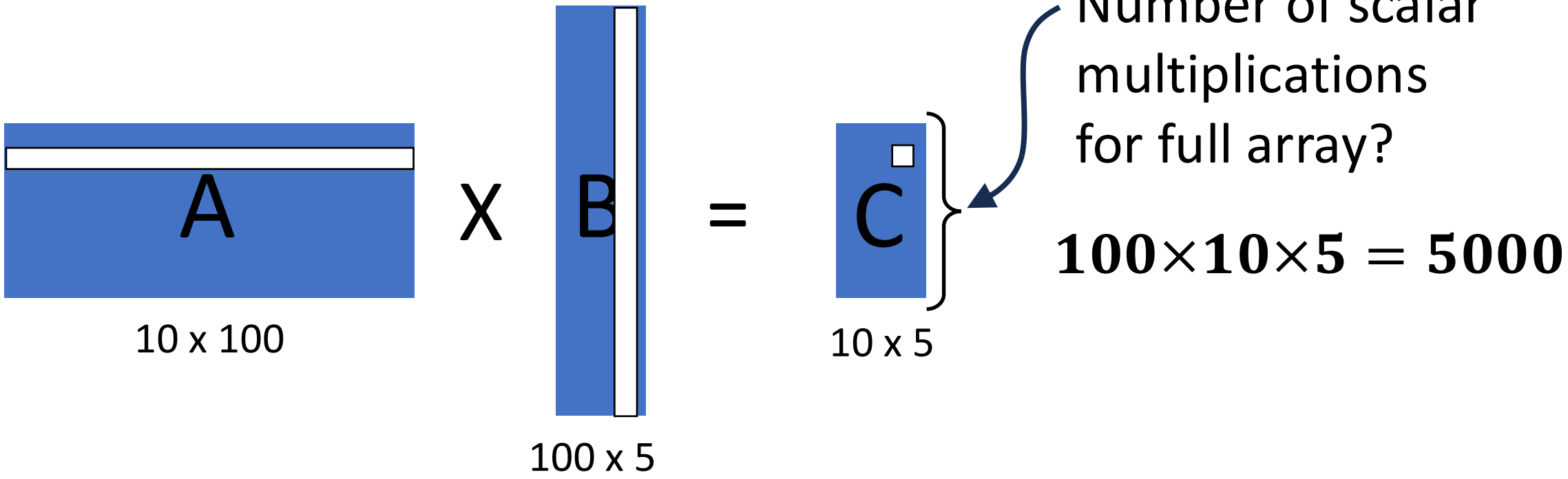
Non-Square Matrix Multiplication Review



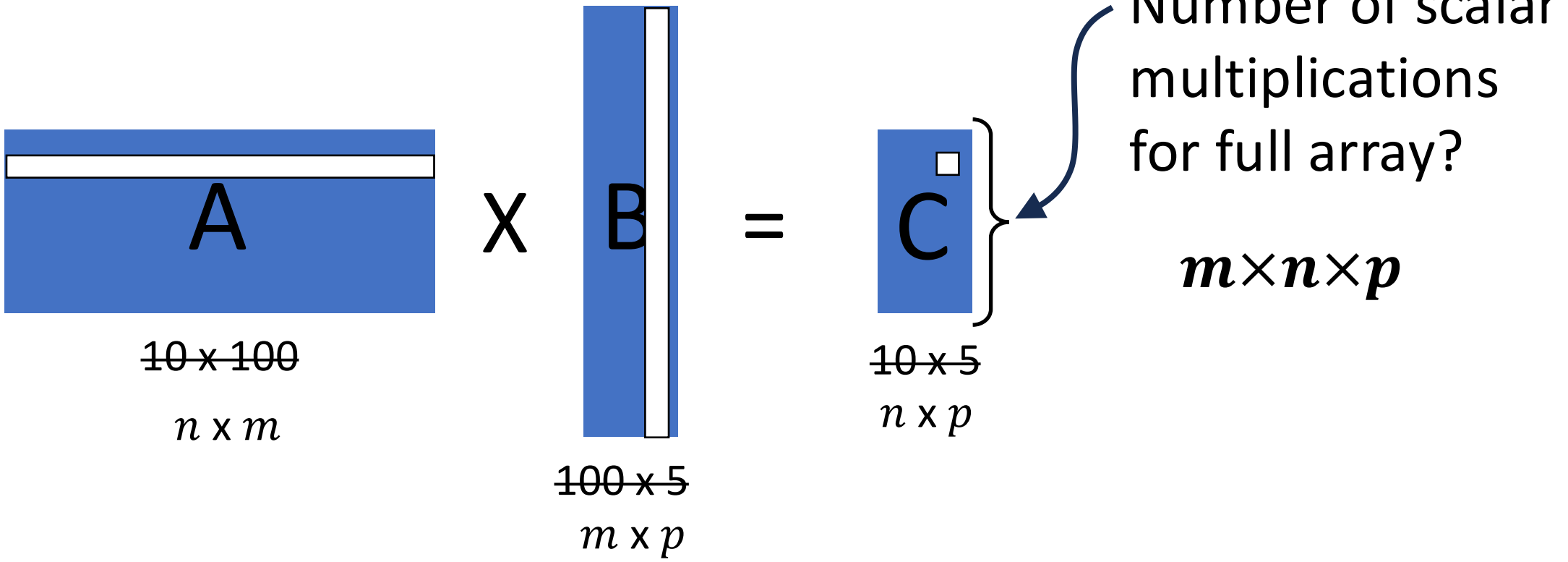
Number of scalar multiplications for single spot?

100

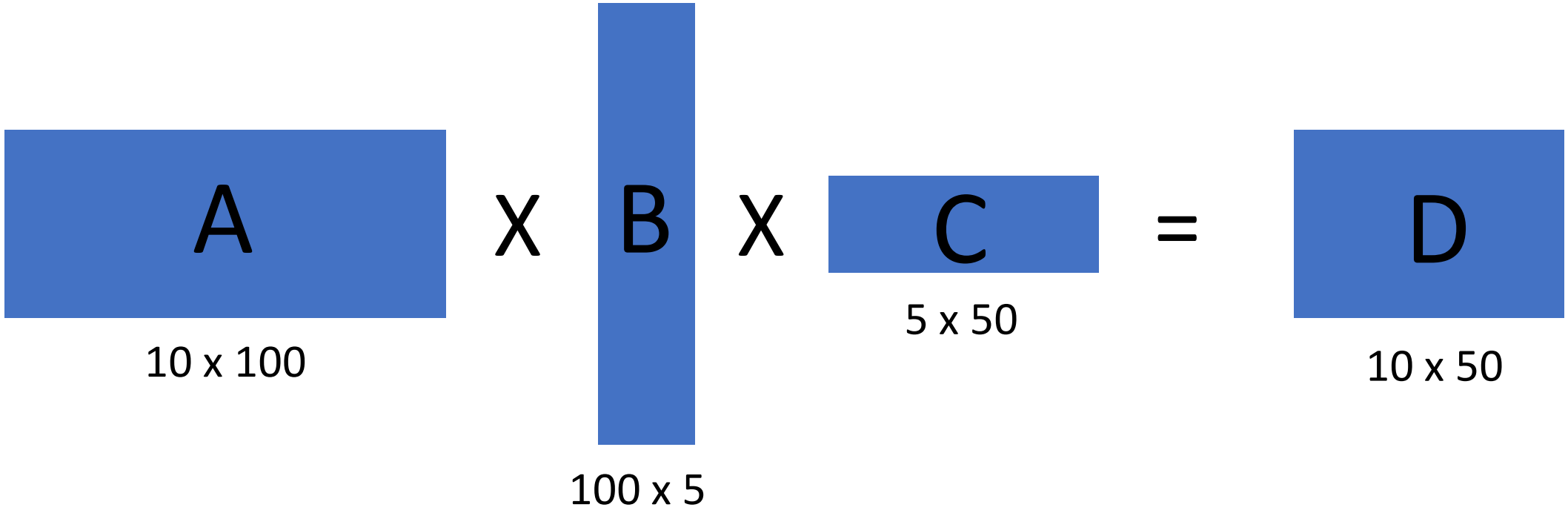
Non-Square Matrix Multiplication Review



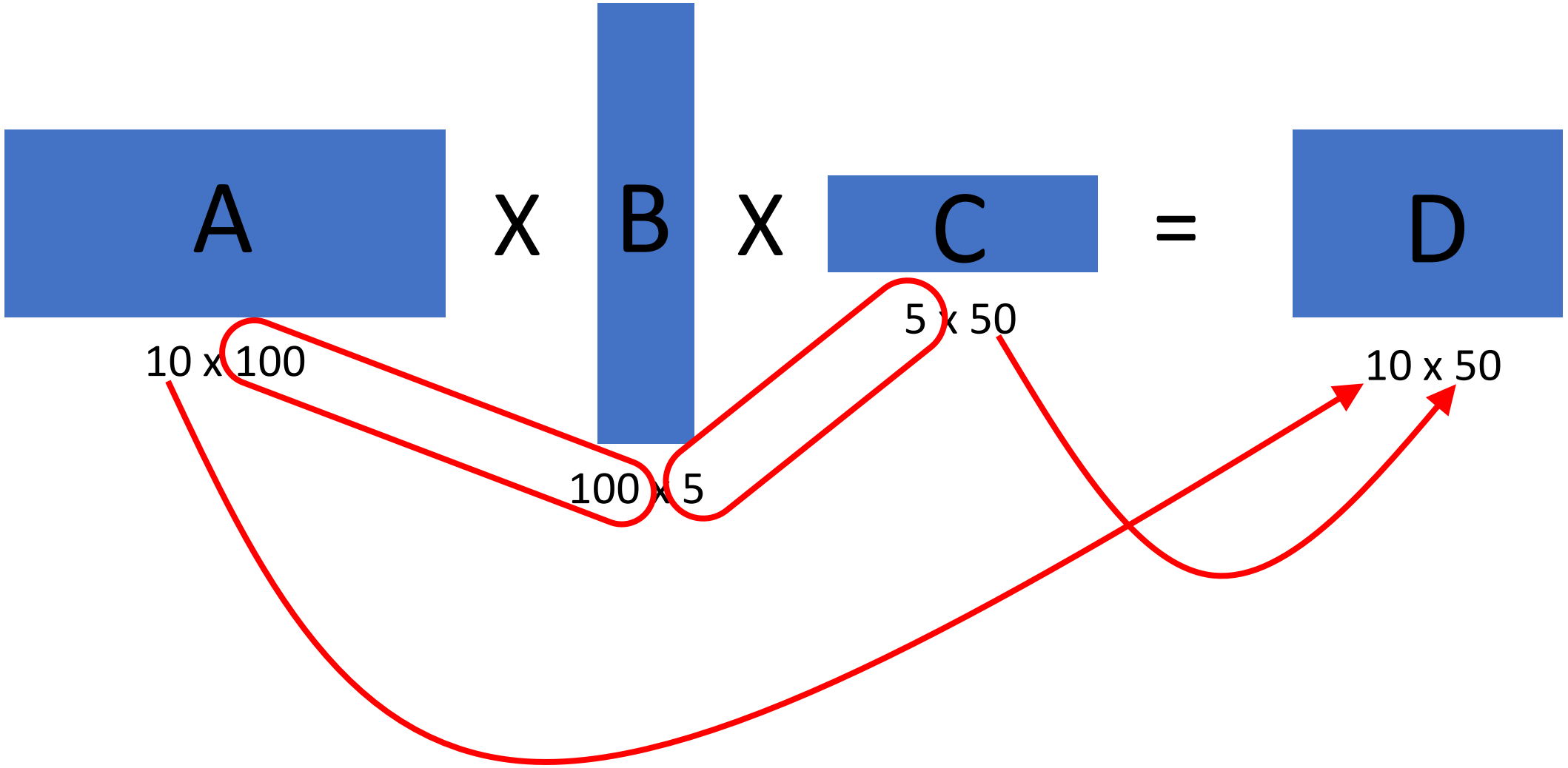
Non-Square Matrix Multiplication Review



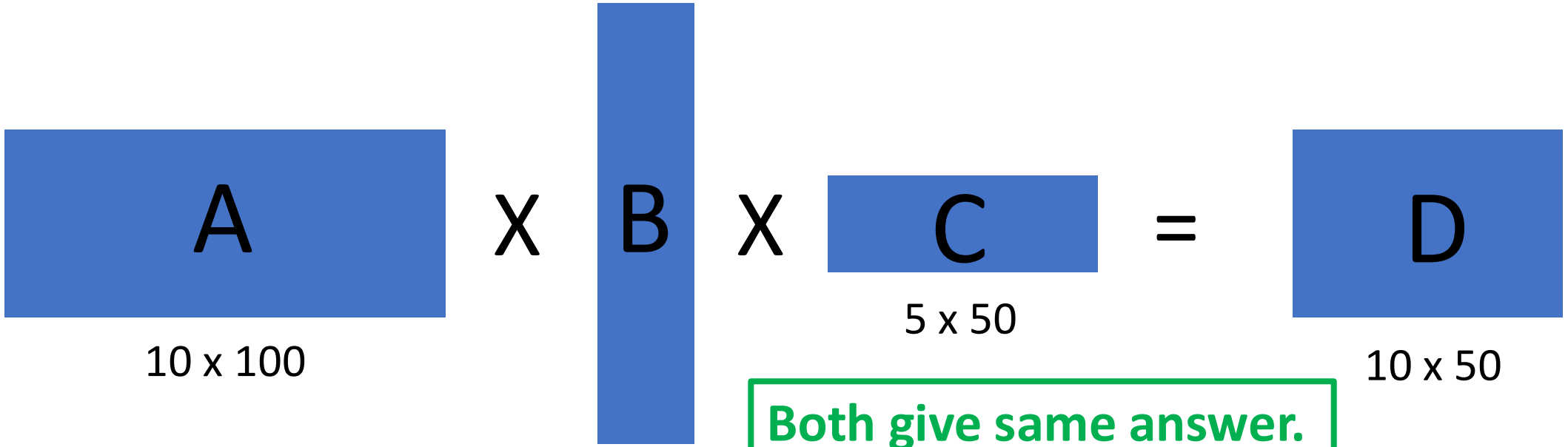
Matrix-Chain Multiplication



Matrix-Chain Multiplication

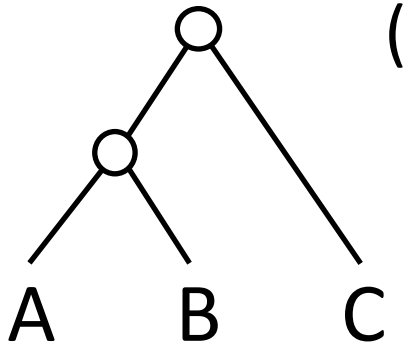


Matrix-Chain Multiplication

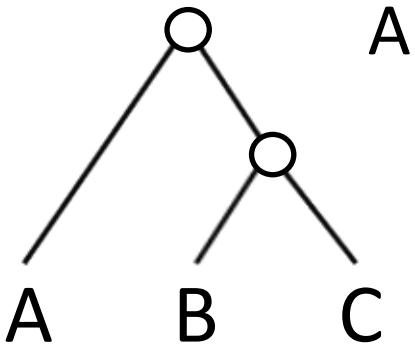


Both give same answer.
Any reason to pick one?

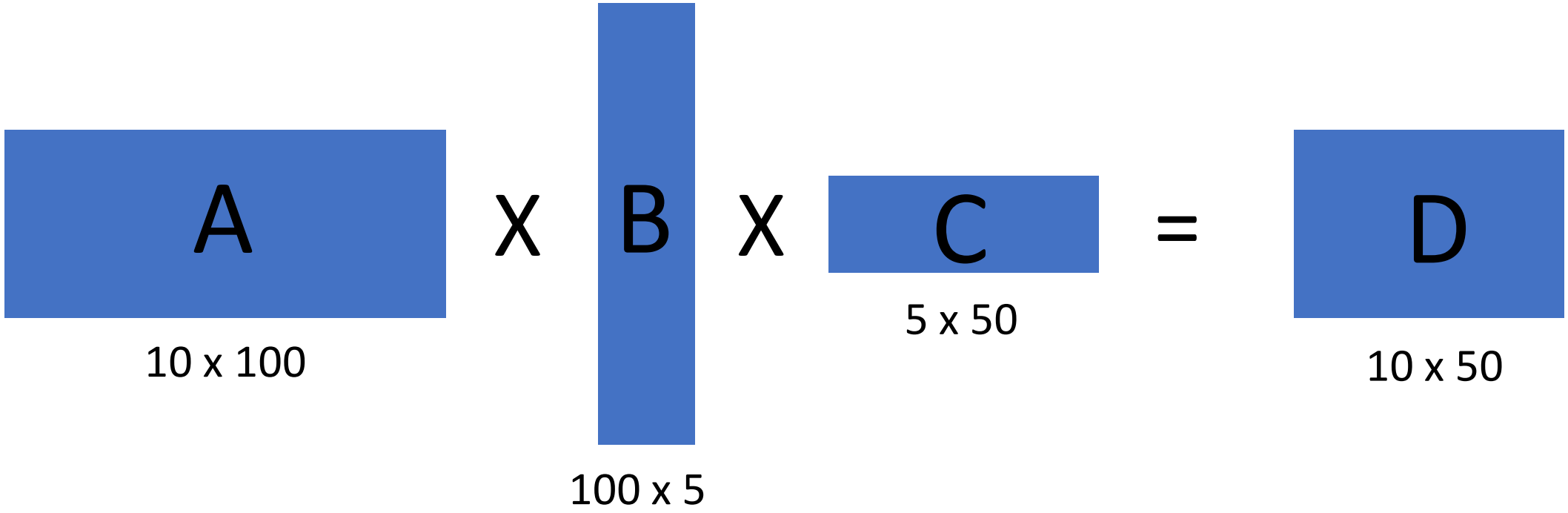
(A x B) x C :

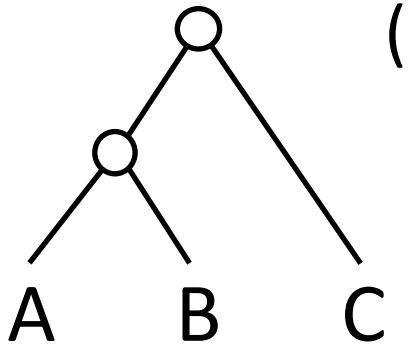


A x (B x C) :



Matrix-Chain Multiplication

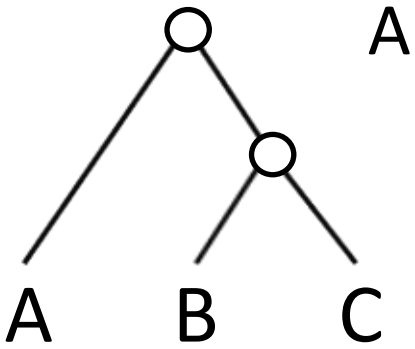




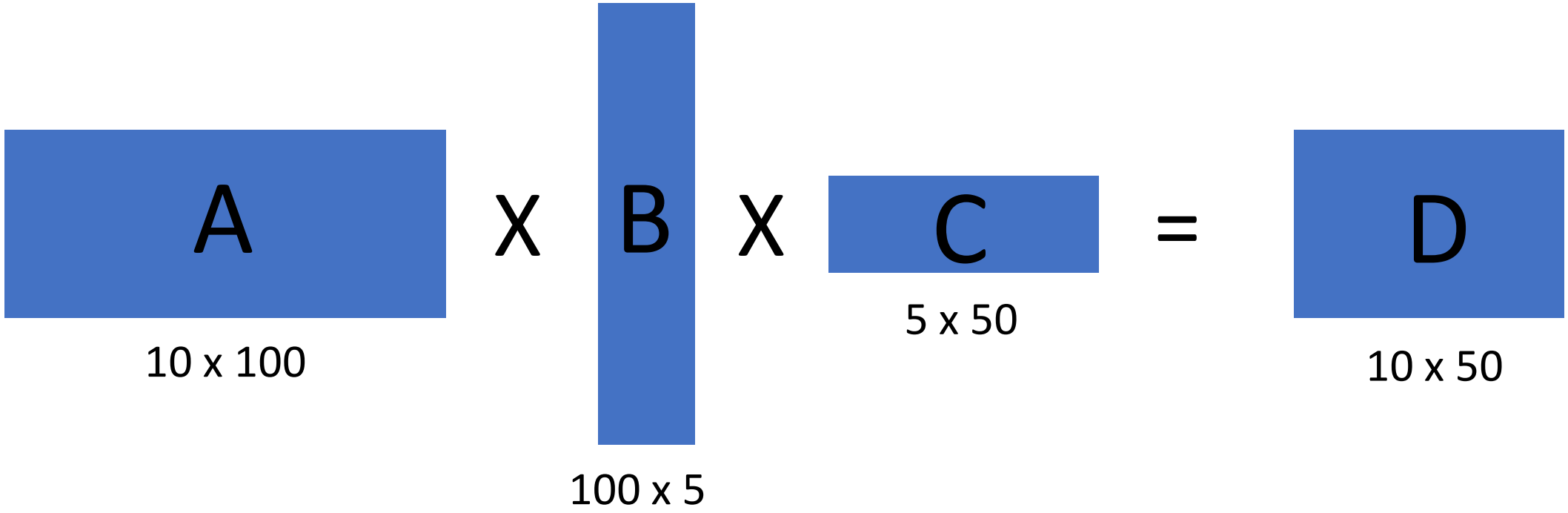
(A x B) x C :

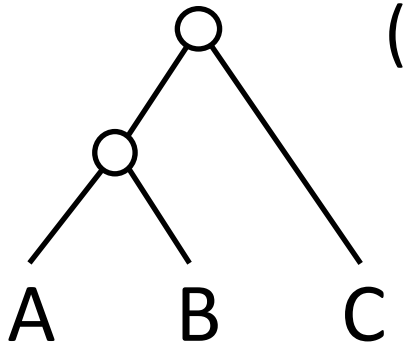
$$100 \times 10 \times 5$$

A x (B x C) :



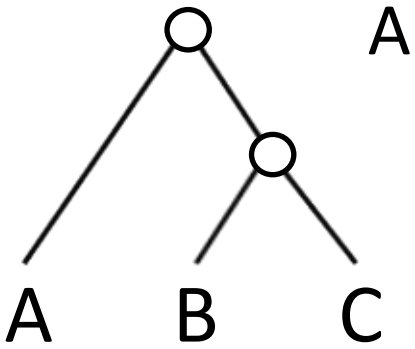
Matrix-Chain Multiplication





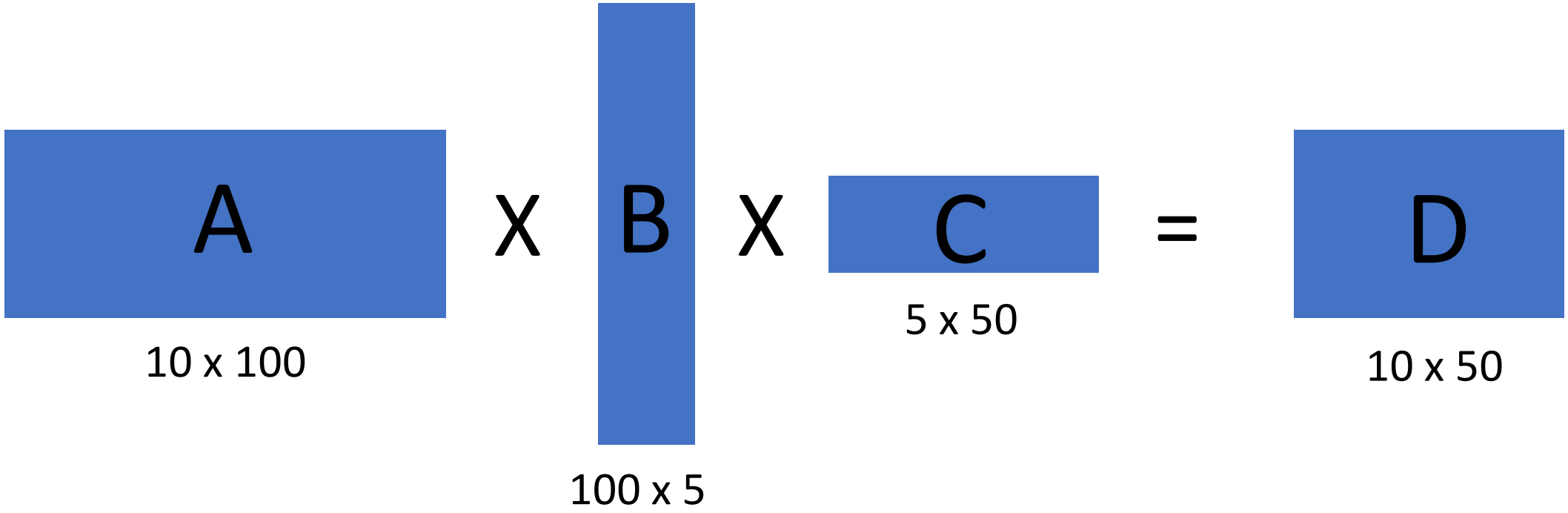
(A x B) x C :

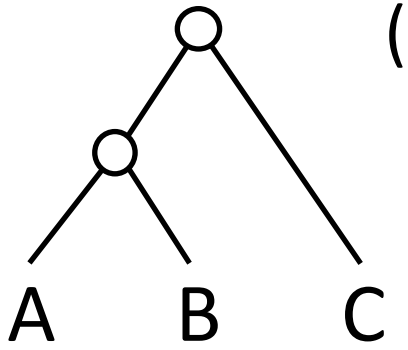
$$100 \times 10 \times 5$$
$$+ 5 \times 10 \times 50$$



A x (B x C) :

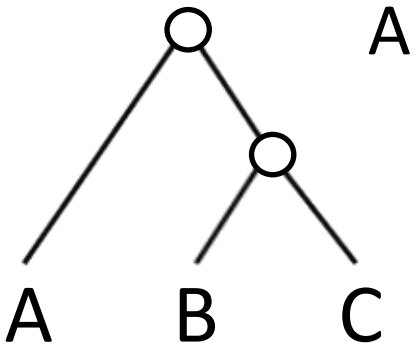
Matrix-Chain Multiplication





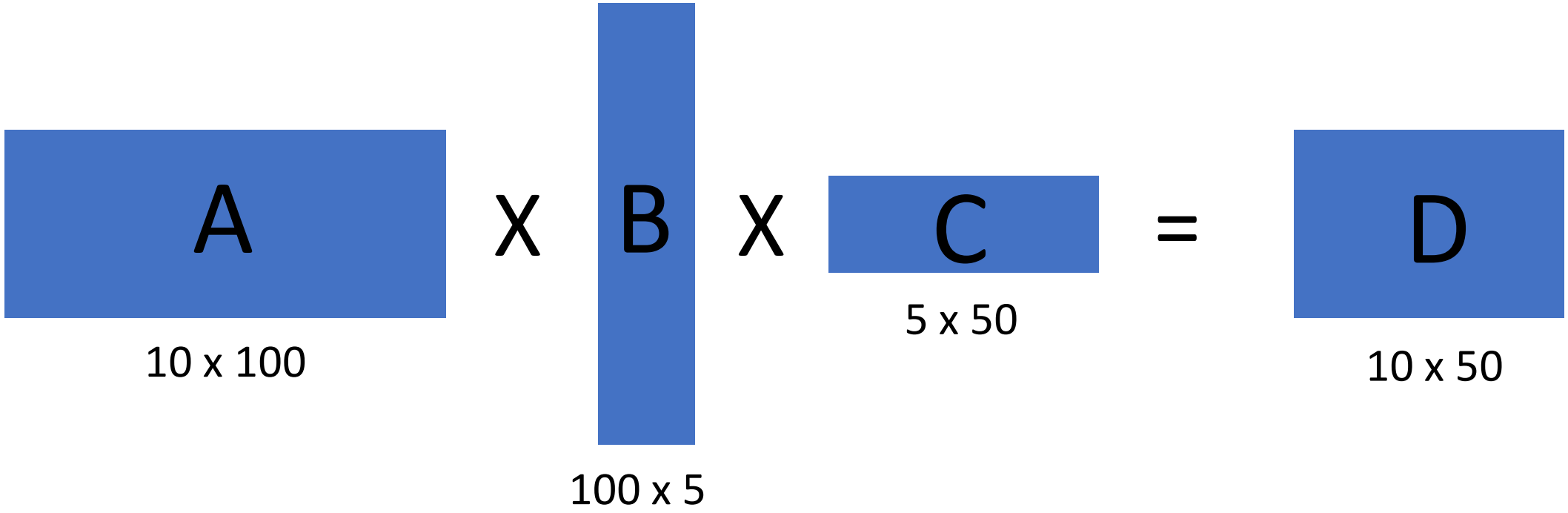
(A x B) x C :

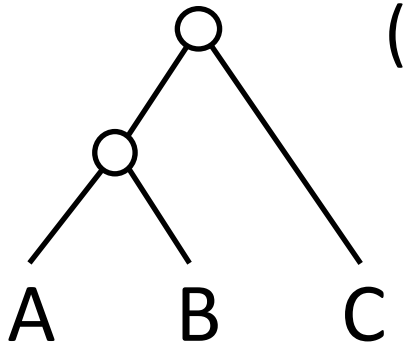
$$100 \times 10 \times 5$$
$$+ 5 \times 10 \times 50$$
$$= 7500$$



A x (B x C) :

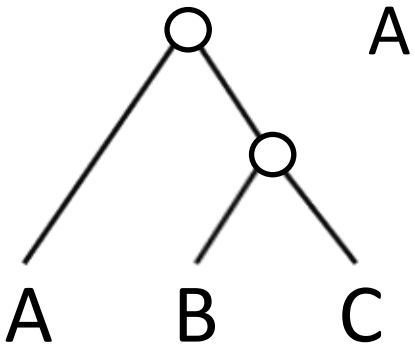
Matrix-Chain Multiplication





(A x B) x C :

$$100\times10\times5$$
$$+5\times10\times50$$
$$= 7500$$



A x (B x C) :

$$5\times100\times50$$
$$+ 100\times10\times50$$
$$= 75000$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times M_3 \times M_4 \times \cdots \times M_{n-4} \times M_{n-3} \times M_{n-2} \times M_{n-1} \times M_n$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times M_3 \times M_4 \times \cdots \times M_{n-4} \times M_{n-3} \times M_{n-2} \times M_{n-1} \times M_n$$

Can we use Dynamic Programming?

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times M_3 \times M_4 \times \cdots \times M_{n-4} \times M_{n-3} \times M_{n-2} \times M_{n-1} \times M_n$$

Is there optimal substructure?

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times M_3 \times M_4 \times \cdots \times M_{n-4} \times M_{n-3} \times M_{n-2} \times M_{n-1} \times M_n$$

Is there optimal substructure?

Given an optimal solution, what can we say about the optimality of a sub solution?

Matrix-Chain Multiplication

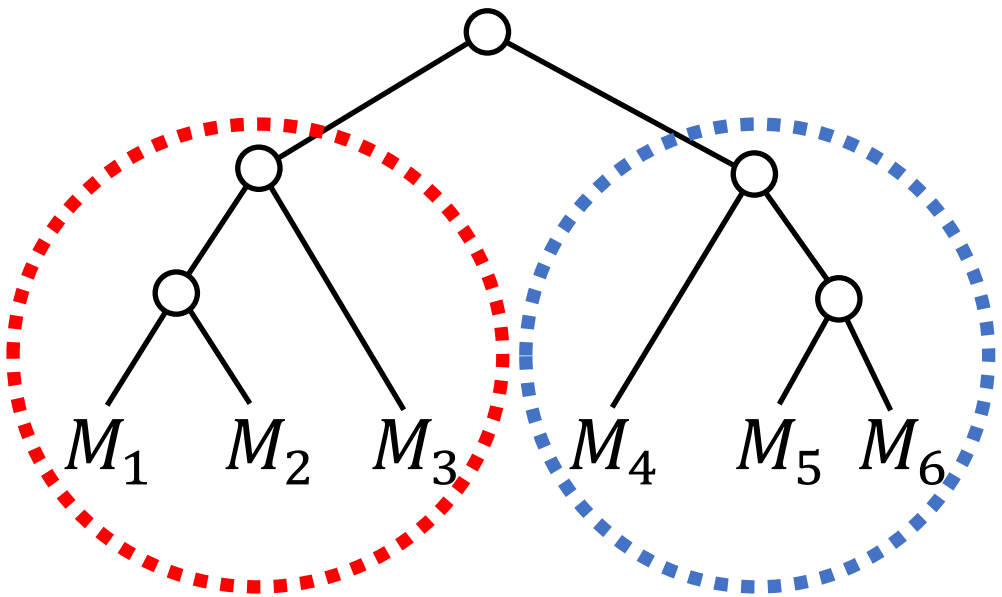
Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$\big((M_1 \times M_2) \times M_3\big) \times \big(M_4 \times (M_5 \times M_6)\big)$$

Is there optimal substructure?

Given an optimal solution, what can we say about the optimality of a sub solution?



Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

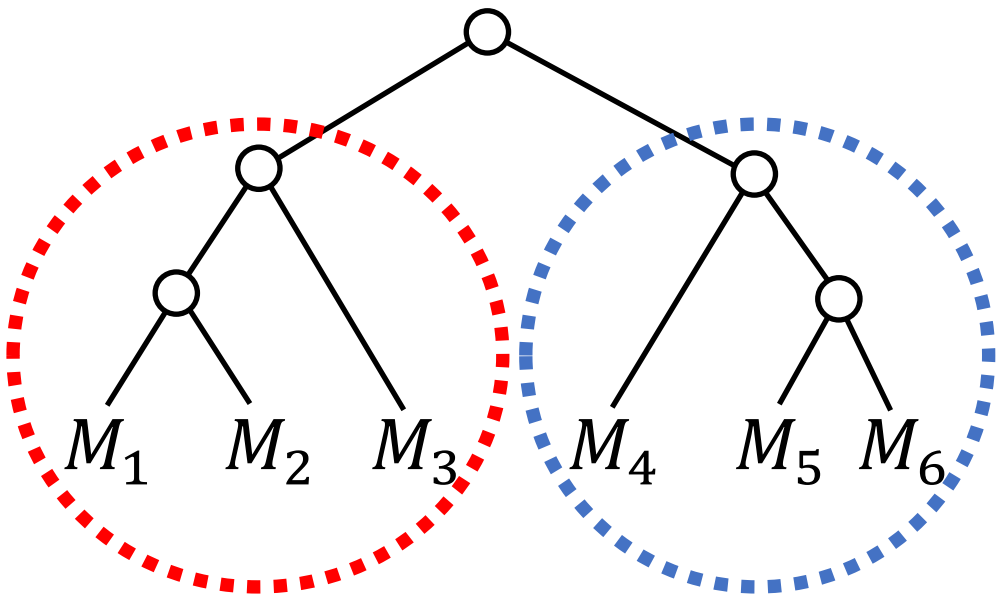
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Is there optimal substructure?

Given an optimal solution, what can we say about the optimality of a sub solution?

An optimal parenthesization must end with multiplying two matrices.



Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

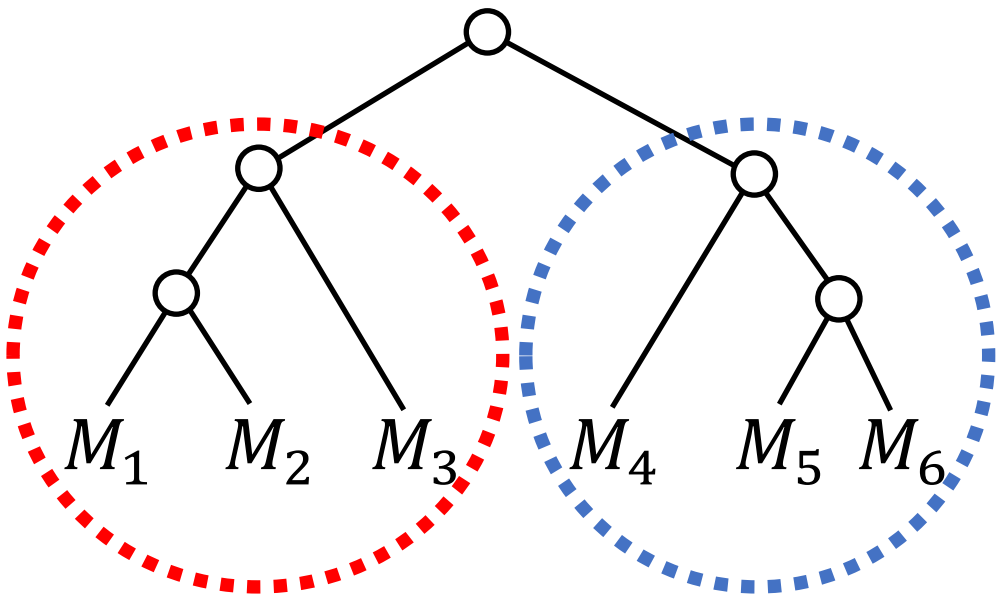
Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$\big((M_1 \times M_2) \times M_3\big) \times \big(M_4 \times (M_5 \times M_6)\big)$$

Is there optimal substructure?

Given an optimal solution, what can we say about the optimality of a sub solution?

An optimal parenthesization must end with multiplying two matrices. The parenthesizations that led to those two matrices must also be optimal.



Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times M_3 \times M_4 \times \cdots \times M_{n-4} \times M_{n-3} \times M_{n-2} \times M_{n-1} \times M_n$$

Let $C[1, n] =$ Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Matrix-Chain Multiplication

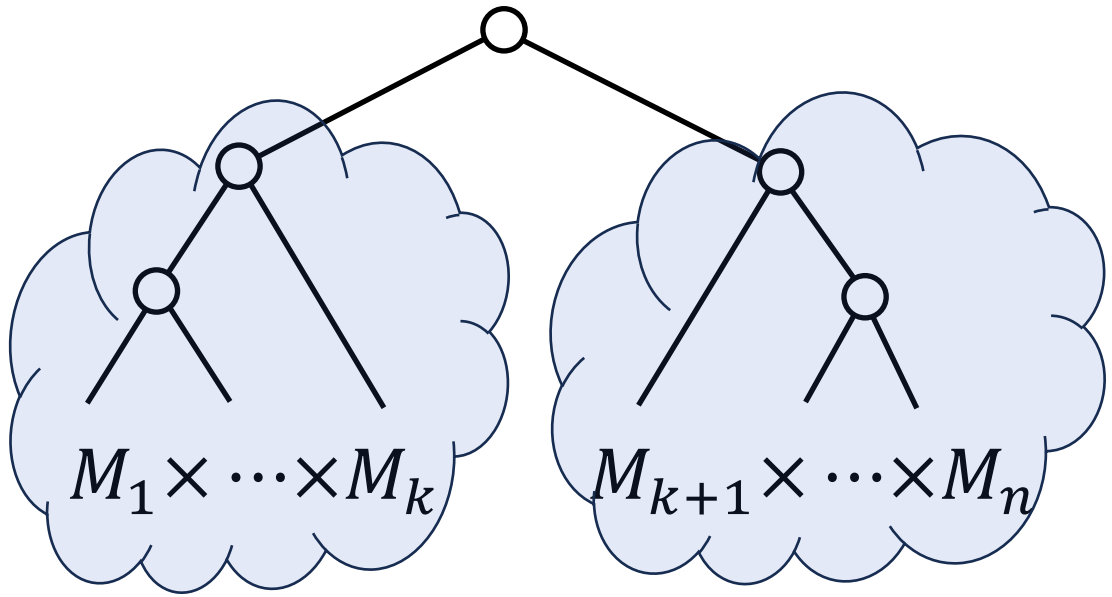
Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.



Matrix-Chain Multiplication

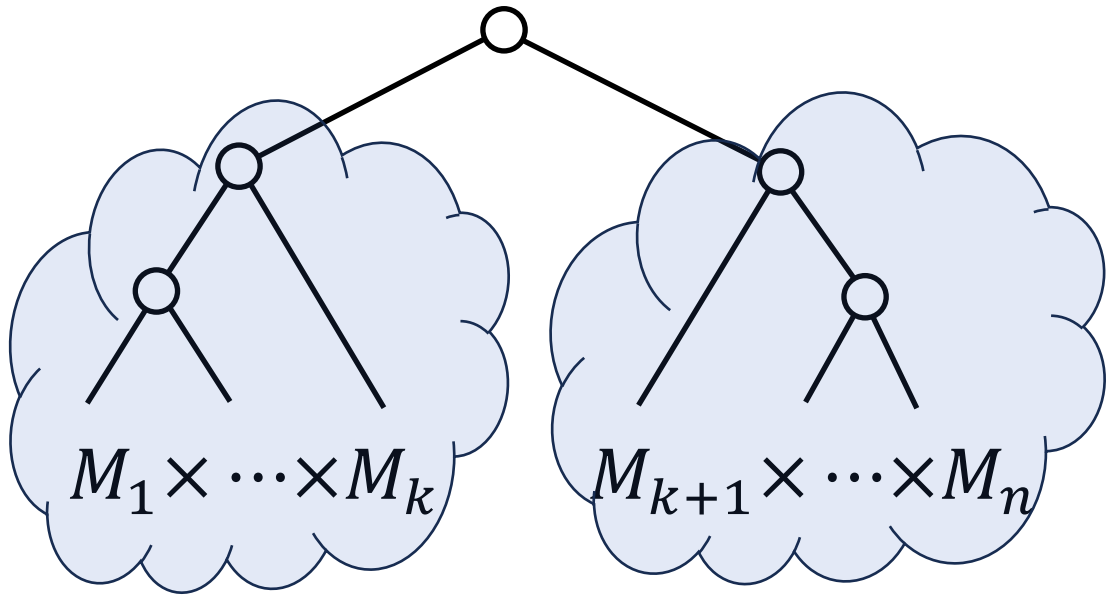
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Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.



$$\boldsymbol{C[1,n]} = \text{??}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \rightarrow M_k$

$$\boldsymbol{C[1, n] =}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \rightarrow M_k$ **Optimal # mults for $M_{k+1} \rightarrow M_n$**

$$\boldsymbol{C[1, n] = \quad + \quad}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

$$C[1,n] = \underbrace{\qquad}_{\text{Optimal \# mults for } M_1 \to M_k} + \underbrace{\qquad}_{\text{Optimal \# mults for } M_{k+1} \to M_n} + \underbrace{\qquad}_{\text{\# mults in final multiplication}}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \rightarrow M_k$

Optimal # mults for $M_{k+1} \rightarrow M_n$

 # mults in final multiplication

$$\boldsymbol{C[1, n] = C[1, k] + \quad + \quad}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1, n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \to M_k$ | **Optimal # mults for $M_{k+1} \to M_n$** | **# mults in final multiplication**

$$\boldsymbol{C[1, n] = C[1, k] + C[k+1, n] +}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

$m_0 \times m_1$ $m_{k-1} \times m_k$ $m_k \times m_{k+1}$ $m_{n-1} \times m_n$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \to M_k$ **Optimal # mults for $M_{k+1} \to M_n$** **# mults in final multiplication**

$$C[1,n] = C[1,k] + C[k+1,n] +$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

$m_0 \times m_1$ $m_{k-1} \times m_k$ $m_k \times m_{k+1}$ $m_{n-1} \times m_n$

Suppose the final multiplication in the optimal solution occurs at k.

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Optimal # mults for $M_1 \to M_k$ + **Optimal # mults for $M_{k+1} \to M_n$** + **# mults in final multiplication**

$$\boldsymbol{C[1,n] = C[1,k] + C[k+1,n] + m_k \times m_0 \times m_n}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Do we know where the final multiplication will occur?

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}}}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Do we know where the final multiplication will occur? No!

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}}}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$(M_1 \times M_2 \times \cdots M_{k-1} \times M_k) \times (M_{k+1} \times \cdots \times M_{n-1} \times M_n)$$

Do we know where the final multiplication will occur? No!
So, what should we do?

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}}}$$

Matrix-Chain Multiplication

Input: n compatible matrices where M_i has dimensions $m_{i-1} \times m_i$

Goal: Multiply matrices M_1 through M_n using the smallest number of scalar multiplications.

$$M_1 \times M_2 \times \cdots M_{k-1} \times M_k \times M_{k+1} \times \cdots \times M_{n-1} \times M_n$$

Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Do we know where the final multiplication will occur? No!
So, what should we do? Check all possibilities…

Optimal # mults for $M_1 \to M_k$ | Optimal # mults for $M_{k+1} \to M_n$ | # mults in final multiplication

$$\boldsymbol{C[1,n] = \overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}}}$$

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Let $C[1,n]$ = Minimum number of scalar multiplications needed for multiplying M_1 through M_n.

Do we know where the final multiplication will occur? No!
So, what should we do? Check all possibilities and pick the best.

$$\boldsymbol{C[1,n] = \min_{1 \le k < n} \left(\overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}} \right)}$$

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This is not generic for any subsequence in the full multiplication.

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This is not generic for any subsequence in the full multiplication.

Let $C[i,j] =$ Minimum number of scalar multiplications needed for multiplying M_i through M_j.

Optimal # mults for $M_1 \to M_k$ | Optimal # mults for $M_{k+1} \to M_n$ | # mults in final multiplication

$$\boldsymbol{C[1,n] = \min_{1 \le k < n} (\overbrace{C[1,k]}^{\text{Optimal \# mults for } M_1 \to M_k} + \overbrace{C[k+1,n]}^{\text{Optimal \# mults for } M_{k+1} \to M_n} + \overbrace{m_k \times m_0 \times m_n}^{\text{\# mults in final multiplication}})}$$

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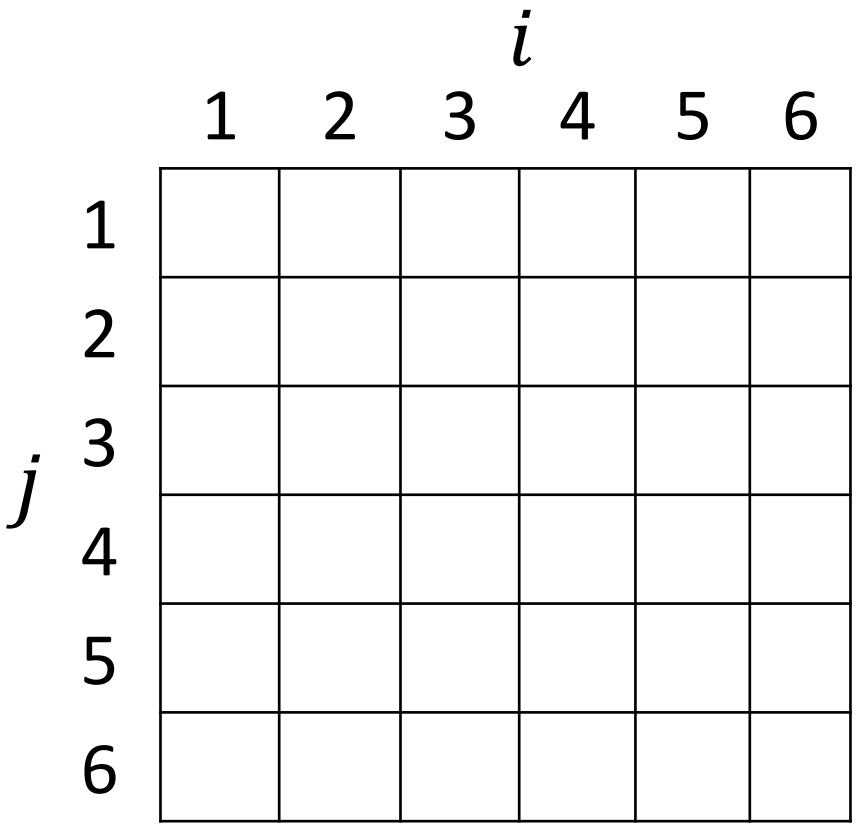
Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$

How are we going to turn this into code?

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$



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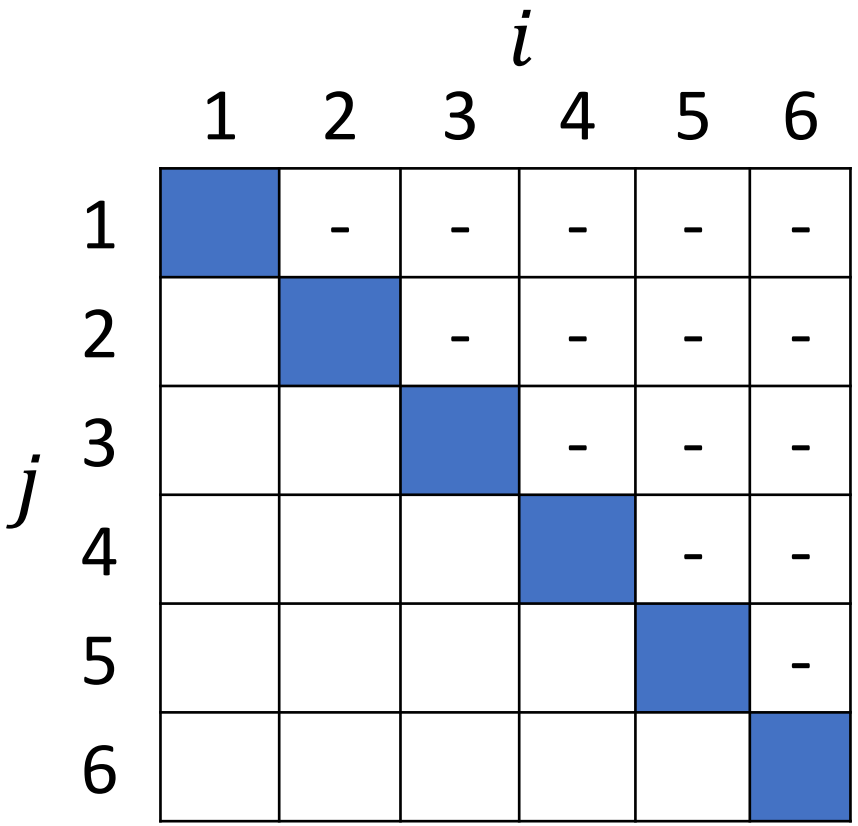
Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$

j \ i	1	2	3	4	5	6
1		-	-	-	-	-
2			-	-	-	-
3				-	-	-
4					-	-
5						-
6						

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$



What can we say about $C[i,i]$?

Let $C[i,j]$ = Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

What can we say about $C[i,i]$?

$$C[i,i] = 0$$

Let $C[i,j]$ = Minimum number of scalar multiplications needed for multiplying $M_i \times M_{i+1} \times \cdots \times M_j$

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Which element of $C[]$ do we want in the end?

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

Which element of $C[]$ do we want in the end?

$$C[1,n]$$

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

What elements of $C[]$ need to be filled out to compute $C[1,6]$?

Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[1,6] = \min_{1 \le k < 6}\left(C[1,k] + C[k+1,6] + ??\right)$$

$$= \min \begin{cases} C[1,1] + C[2,6] + ?? & k = 1 \\ C[1,2] + C[3,6] + ?? & k = 2 \\ C[1,3] + C[4,6] + ?? & k = 3 \\ C[1,4] + C[5,6] + ?? & k = 4 \\ C[1,5] + C[6,6] + ?? & k = 5 \end{cases}$$

Matrix-Chain Multiplication

$$M_1 \times M_2 \times M_3 \times M_4 \times M_5 \times M_6$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

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Matrix-Chain Multiplication

$$(M_1) \times (M_2 \times M_3 \times M_4 \times M_5 \times M_6)$$

$$k = 1$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

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Matrix-Chain Multiplication

$$(M_1 \times M_2) \times (M_3 \times M_4 \times M_5 \times M_6)$$

$$\boldsymbol{k = 2}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[1,6] = \min_{1 \le k < 6} (C[1,k] + C[k+1,6] + ??)$$

$$= \min \begin{cases} C[1,1] + C[2,6] + ?? & k = 1 \\ C[1,2] + C[3,6] + ?? & k = 2 \\ C[1,3] + C[4,6] + ?? & k = 3 \\ C[1,4] + C[5,6] + ?? & k = 4 \\ C[1,5] + C[6,6] + ?? & k = 5 \end{cases}$$

Matrix-Chain Multiplication

$$(M_1 \times M_2 \times M_3) \times (M_4 \times M_5 \times M_6)$$

$$\boldsymbol{k = 3}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[1,6] = \min_{1 \le k < 6} (C[1,k] + C[k+1,6] + ??)$$

$$= \min \begin{cases} C[1,1] + C[2,6] + ?? & k = 1 \\ C[1,2] + C[3,6] + ?? & k = 2 \\ C[1,3] + C[4,6] + ?? & k = 3 \\ C[1,4] + C[5,6] + ?? & k = 4 \\ C[1,5] + C[6,6] + ?? & k = 5 \end{cases}$$

Matrix-Chain Multiplication

$$(M_1 \times M_2 \times M_3 \times M_4) \times (M_5 \times M_6)$$

$$\boldsymbol{k = 4}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[1,6] = \min_{1 \le k < 6} (C[1,k] + C[k+1,6] + ??)$$

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Matrix-Chain Multiplication

$$(M_1 \times M_2 \times M_3 \times M_4 \times M_5) \times (M_6)$$

$$\boldsymbol{k = 5}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

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Matrix-Chain Multiplication

$$\boldsymbol{C[i,j] = \min_{i \le k < j}\left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j\right)}$$

j \ i	1	2	3	4	5	6
1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
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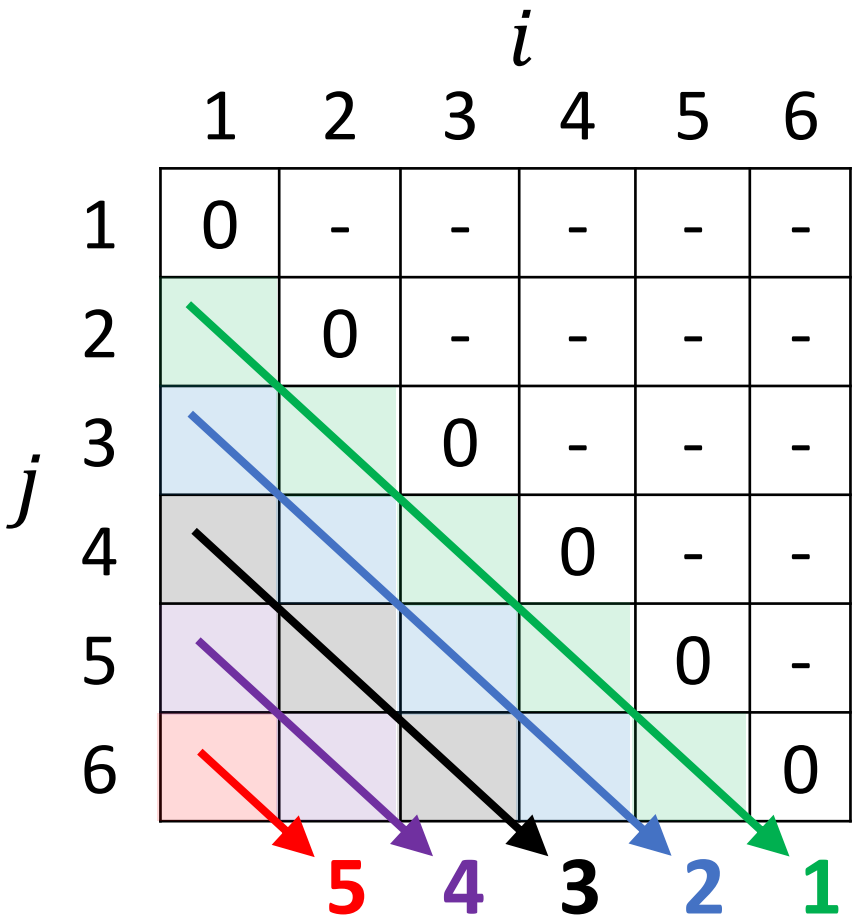
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1	0	-	-	-	-	-
2		0	-	-	-	-
3			0	-	-	-
4				0	-	-
5					0	-
6						0

$$C[3,5] = \min_{3 \le k < 5}\left(C[3,k] + C[k+1,5] + ??\right)$$

$$= \min \begin{cases} C[3,3] + C[4,5] + ?? & k = 1 \\ C[3,4] + C[5,5] + ?? & k = 2 \end{cases}$$

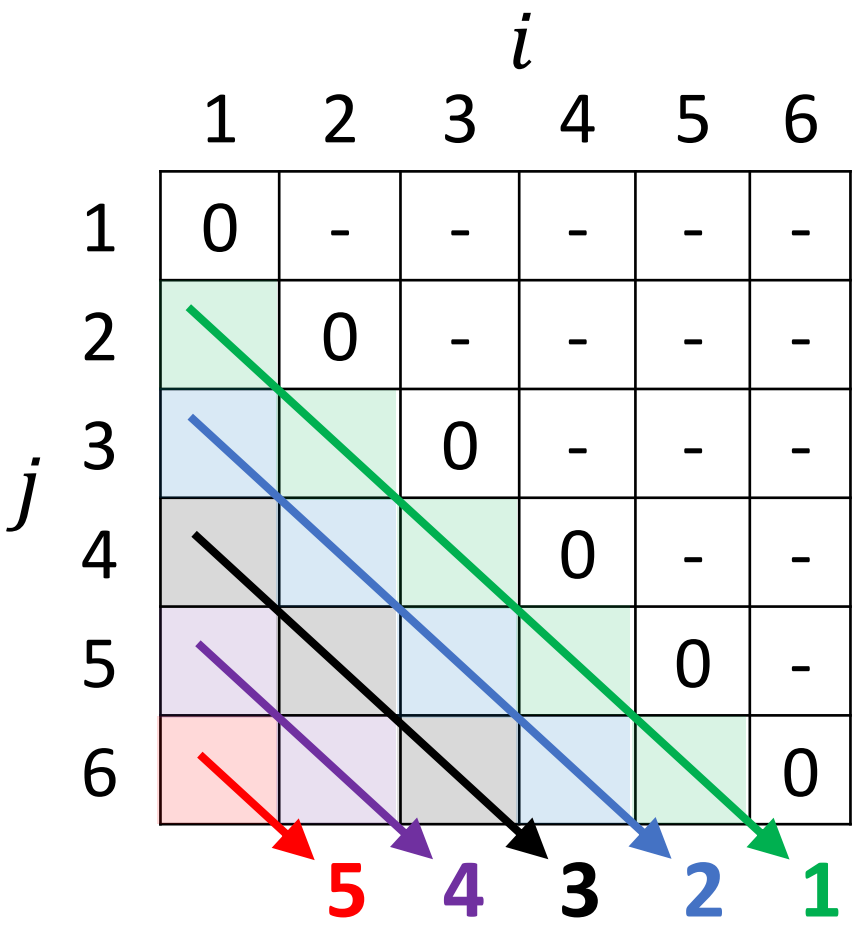
Matrix-Chain Multiplication

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Matrix-Chain Multiplication

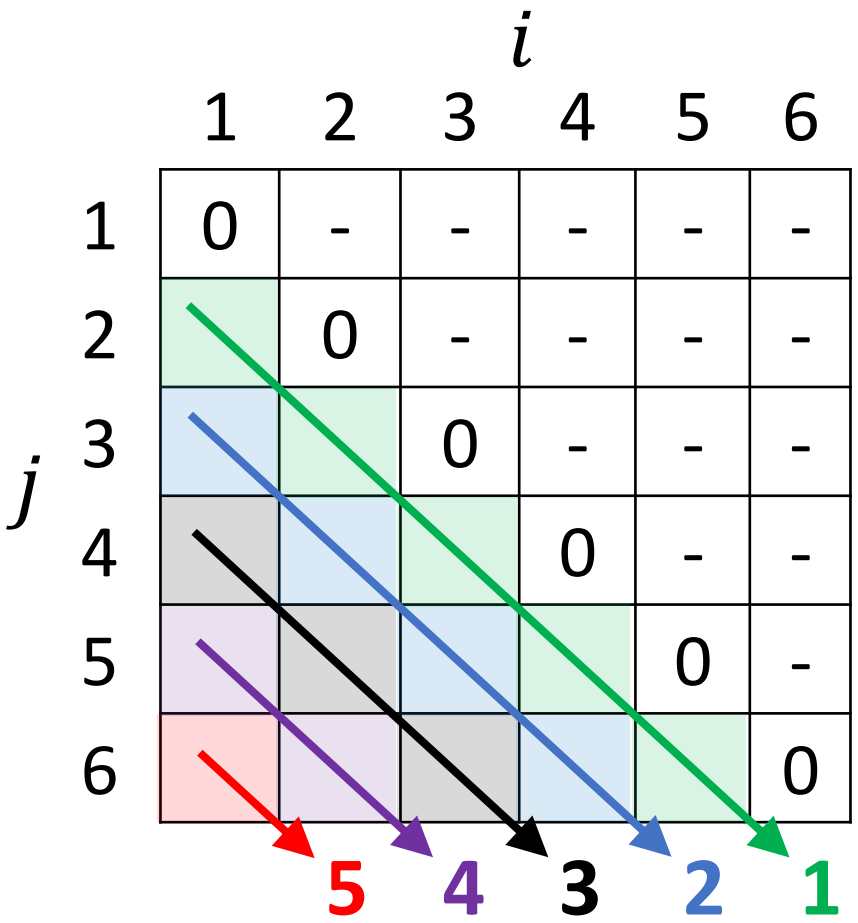
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```
for s = 1,...,n - 1
  for i = 1,...,n - s
    j = i + s
```

Matrix-Chain Multiplication

$$C[i,j] = \min_{i \le k < j} \left(C[i,k] + C[k+1,j] + m_k \times m_{i-1} \times m_j \right)$$




```
for s = 1,...,n - 1
  for i = 1,...,n - s
    j = i + s
    min = ∞
    for k = i,...,j - 1
      a = C[i,k]+C[k+1,j]+m_k m_{i-1} m_j
      if a < min
        min = a
    C[i,j] = min
return C[1,n]
```

Running Time:
$O(n^3)$